

On Some Autotopisms Of Non-Steiner Central Loops ^{*†}

T. G. Jaiyéolá[‡]

Department of Mathematics,
Obafemi Awolowo University,
Ile Ife 220005, Nigeria.
jaiyeolatemitope@yahoo.com
tjayeola@oauife.edu.ng

J. O. Adéníran

Department of Mathematics,
University of Abeokuta,
Abeokuta 110101, Nigeria.
ekenedilichineke@yahoo.com
adeniranoj@unaab.edu.ng

Abstract

An algebraic process for the construction of an autotopism for a non-Steiner C-loop is described and this is demonstrated with an example using a known finite C-loop. In every C-loop, two of its parastrophes are equivalent(equal) to it, if and only if both the first and second components of the constructed autotopism and its inverse autotopism are equal to the identity map. Hence, the other three parastrophes are equivalent(equal) to the C-loop. It is proved that the set of autotopisms that prevent a C-loop from being a Steiner loop forms a Steiner triple system.

1 Introduction

LC-loops, RC-loops and C-loops are loops that satisfy the identities

$$(xx)(yz) = (x(xy))z, (zy)(xx) = z((yx)x) \text{ and } x(y(yz)) = ((xy)y)z \text{ respectively.}$$

These three types of loops are collectively called central loops. In the theory of loops, central loops are some of the least studied loops. They have been studied by Phillips and Vojtěchovský [20], [18], [19], Kinyon et. al. [15], [13], [14], Ramamurthi and Solarin [21], Fenyves [9] and Beg [2], [3]. The difficulty in studying them is as a result of the nature of the identities defining them when compared with other Bol-Moufang identities. It can be noticed that in the aforementioned LC identity, the two x variables are consecutively positioned and neither y nor z is between them. A similarly observation is true in the other two identities(i.e the RC and C identities). But this observation is not true in the identities defining Bol loops,

^{*}2000 Mathematics Subject Classification. Primary 20N05 ; Secondary 08A05

[†]**Keywords and Phrases :** C-loops, Steiner loops, autotopisms, parastrophes

[‡]All correspondence to be addressed to this author.

Moufang loops and extra loops. Fenyves [9] gave three equivalent identities that define LC-loops, three equivalent identities that define RC-loops and only one identity that defines C-loops. But recently, Phillips and Vojtěchovský [18], [19] gave four equivalent identities that define LC-loops and four equivalent identities that define RC-loops. Three of the four identities given by Phillips and Vojtěchovský are the same as the three already given by Fenyves.

Their basic properties are found in [20], [21], [9] and [7]. The left and right translation maps on the loop (L, \cdot) denoted by $L_x : L \rightarrow L$ and $R_x : L \rightarrow L$ and defined as $yL_x = xy$ and $yR_x = yx$ respectively $\forall x, y \in L$ are bijections. L is said to be left alternative and right alternative if

$$x \cdot xy = x^2y \text{ and } yx \cdot x = yx^2 \text{ respectively } \forall x, y \in L$$

Thus, L is said to be alternative if it is both left and right alternative. L is called a Steiner loop if and only if

$$x^2 = e, \quad yx \cdot x = y \text{ and } xy = yx \quad \forall x, y \in L.$$

The set $S(L, \cdot)$ of all bijections in a loop (L, \cdot) forms a group called the permutation group of the loop (L, \cdot) . The triple (U, V, W) such that $U, V, W \in S(L, \cdot)$ is called an autotopism of L if and only if

$$xU \cdot yV = (x \cdot y)W \quad \forall x, y \in L.$$

The group of autotopisms of L is denoted by $Aut(L, \cdot)$.

An algebraic process for the construction of an autotopism for a non-Steiner C-loop is described and this is demonstrated with an example using a known finite C-loop. In every C-loop, two of its parastrophes are equivalent(equal) to it, if and only if both the first and second components of the constructed autotopism and its inverse autotopism are equal to the identity map. Hence, the other three parastrophes are equivalent(equal) to the C-loop. It is proved that the set of autotopisms that prevent a C-loop from being a Steiner loop forms a Steiner triple system.

Definition 1.1 ([7], Page 65)

Let (L, θ) be a quasigroup. The 5 parastrophes or conjugates or adjugates of (L, θ) are quasigroups whose binary operations θ^* , θ^{-1} , $^{-1}\theta$, $(\theta^{-1})^*$, $(^{-1}\theta)^*$ defined on L are given by :

(a)

$$(L, \theta^*) : y\theta^*x = z \Leftrightarrow x\theta y = z \quad \forall x, y, z \in L.$$

(b)

$$(L, \theta^{-1}) : x\theta^{-1}z = y \Leftrightarrow x\theta y = z \quad \forall x, y, z \in L.$$

(c)

$$(L, ^{-1}\theta) : z^{-1}\theta y = x \Leftrightarrow x\theta y = z \quad \forall x, y, z \in L.$$

(d)

$$\left(L, (\theta^{-1})^* \right) : z(\theta^{-1})^* x = y \Leftrightarrow x\theta y = z \quad \forall x, y, z \in L.$$

(e)

$$\left(L, (-^1\theta)^* \right) : y(-^1\theta)^* z = x \Leftrightarrow x\theta y = z \quad \forall x, y, z \in L.$$

Remark 1.1 As it can be seen in Definition 1.1, every quasigroup (L, \cdot) belongs to a set of 6 quasigroups, called adjugates by Fisher and Yates [10], conjugates by Stein [25], [24] and Belousov [4] and parastrophes by Sade [22]. They have been studied by Artzy [1], Lindner and Steedley [16] and a detailed study on them can be found in [17], [6] and [7]. The most recent studies of the parastrophes of a quasigroup(loop) are by Duplak [8], Shchukin and Gushan [23], Frank, Bennett and Zhang [11].

Definition 1.2 A Steiner triple system (S.T.S. for short) $(Q, 3)$ on a set Q is a set of unordered triples $\{a, b, c\} \in (Q, 3)$ such that

(i) a, b, c are distinct elements of Q ,

(ii) to any $a, b \in Q$ such that $a \neq b$ there exists a unique triple $\{a, b, c\} \in (Q, 3)$.

Remark 1.2 It is proved in [7] and stated in [17] that if $|(Q, 3)| = r$ where $(Q, 3)$ is as defined in Definition 1.2, then, $r \equiv 1 \pmod{6}$ or $r \equiv 3 \pmod{6}$.

Definition 1.3 Let $(G, *)$ and (H, \star) be two distinct groupoids. $(G, *)$ and (H, \star) are said to be equivalent or equal, written as $(G, *) \equiv (H, \star)$ or $(G, *) = (H, \star)$ respectively, if $G = H$ and $'* = \star'$. That is, $(G, *)$ and (H, \star) are the same.

2 Autotopisms of Central Loops

Theorem 2.1 A loop L is an LC-loop $\Leftrightarrow (L_x^2, I, L_x^2) \in \text{Aut}(L) \quad \forall x \in L$.

Proof

Let L be an LC-loop $\Leftrightarrow (x \cdot xy)z = (xx)(yz) \Leftrightarrow (x \cdot xy)z = x(x \cdot yz)$ by [7] $\Leftrightarrow (L_x^2, I, L_x^2) \in \text{Aut}(L) \quad \forall x \in L$.

Theorem 2.2 A loop L is an RC-loop $\Leftrightarrow (I, R_x^2, R_x^2) \in \text{Aut}(L) \quad \forall x \in L$.

Proof

Let L be an RC-loop, then $z(yx \cdot x) = zy \cdot xx \Leftrightarrow y(yx \cdot x) = (zy \cdot x)x$ by [7] $\Leftrightarrow (I, R_x^2, R_x^2) \in \text{Aut}(L) \quad \forall x \in L$.

Lemma 2.1 *Let L be a C-loop. Then for each $(A, B, C) \in \text{Aut}(L, \cdot)$, there exists a unique pair of $(S_1, T_1, \mathcal{R}_1), (S_2, T_2, \mathcal{R}_2) \in \text{Aut}(L, \cdot)$ for each $x \in L$ such that $L_x^2 = S_2^{-1}S_1, R_x^2 = T_1^{-1}T_2, R_x^{-2}L_x^2 = \mathcal{R}_2^{-1}\mathcal{R}_1, \mathcal{R}_1^{-1}\mathcal{R}_2T_2^{-1}T_1S_2^{-1}S_1 = I$.*

Proof

By Theorem 2.1 and Theorem 2.2;

$$(S_1, T_1, \mathcal{R}_1) = (A, B, C)(L_x^2, I, L_x^2) \in \text{Aut}(L)$$

$$(S_2, T_2, \mathcal{R}_2) = (A, B, C)(I, R_x^2, R_x^2) \in \text{Aut}(L).$$

Hence, the conditions hold although the identities do not depend on (A, B, C) , but the uniqueness does.

Theorem 2.3 *Let L be a C-loop and let there exist a unique pair of autotopisms $(S_1, T_1, \mathcal{R}_1), (S_2, T_2, \mathcal{R}_2)$ such that the conditions $L_x^2 = S_2^{-1}S_1, R_x^2 = T_1^{-1}T_2$ and $R_x^{-2}L_x^2 = \mathcal{R}_2^{-1}\mathcal{R}_1$ hold for each fixed $x \in L$. If $\alpha_1 = S_1^{-1}, \alpha_2 = S_2^{-1}, \beta_1 = T_1^{-1}, \beta_2 = T_2^{-1}, \gamma_1 = \mathcal{R}_1^{-1}$ and $\gamma_2 = \mathcal{R}_2^{-1}$, then:*

$$(x^2y)\alpha_1 = y\alpha_2 \quad , \quad e\alpha_1 = x^{-2}\alpha_2 \quad , \quad x^{m+2}\alpha_1 = x^m\alpha_2$$

$$(yx^2)\beta_2 = y\beta_1 \quad , \quad e\beta_2 = x^{-2}\beta_1 \quad , \quad x^{m+2}\beta_2 = x^m\beta_1$$

$$(x^2yx^{-2})\gamma_1 = y\gamma_2 \quad , \quad e\gamma_1 = e\gamma_2 \quad , \quad x^m\gamma_1 = x^m\gamma_2$$

for all $m \in \mathbb{Z}$ and $x, y \in L$.

Proof

From Lemma 2.1:

$$L_x^2 = S_2^{-1}S_1, R_x^2 = T_1^{-1}T_2, R_x^{-2}L_x^2 = \mathcal{R}_2^{-1}\mathcal{R}_1.$$

Keeping in mind that a C-loop is power associative and nuclear square, we have the following proofs.

$$1. \quad L_x^2 = S_2^{-1}S_1 \Rightarrow yL_x^2 = yS_2^{-1}S_1 \quad \forall y \in L \Rightarrow yL_{x^2} = yS_2^{-1}S_1 \Rightarrow x^2y = yS_2^{-1}S_1 \Rightarrow (x^2y)S_1^{-1} = yS_2^{-1} \Rightarrow x^2y\alpha_1 = y\alpha_2.$$

$$\text{Let } y = x^{-2} ; \quad x^2x^{-2}\alpha_1 = x^{-2}\alpha_2 \Rightarrow e\alpha_1 = x^{-2}\alpha_2.$$

$$\text{Let } y = x^m ; \quad x^2y\alpha_1 = x^2x^m\alpha_1 = x^m\alpha_2 \Rightarrow x^{m+2}\alpha_1 = x^m\alpha_2.$$

$$2. \quad R_x^2 = T_1^{-1}T_2 \Rightarrow yR_x^2 = yT_1^{-1}T_2 \quad \forall y \in L \Rightarrow yx^2 = yT_1^{-1}T_2 \Rightarrow yx^2T_2^{-1} = yT_1^{-1} \Rightarrow yx^2\beta = y\beta_1.$$

$$\text{Let } y = x^{-2} ; \quad yx^2\beta_2 = x^{-2}x^2\beta_2 = e\beta_2 = x^{-2}\beta_1 \Rightarrow e\beta_2 = x^{-2}\beta_1.$$

$$\text{Let } y = x^m ; \quad yx^2\beta_2 = x^m x^2\beta_2 = x^{m+2}\beta_2 = x^m\beta_1 \Rightarrow x^{m+2}\beta_2 = x^m\beta_1.$$

$$3. \quad R_x^{-2}L_x^2 = \mathcal{R}_2^{-1}\mathcal{R}_1 \Rightarrow yR_x^{-2}L_x^2 = y\mathcal{R}_2^{-1}\mathcal{R}_1 \quad \forall y \in L \Rightarrow x^2yx^{-2} = y\mathcal{R}_2^{-1}\mathcal{R}_1 \Rightarrow (x^2yx^{-2})\mathcal{R}_1^{-1} = y\mathcal{R}_2^{-1} \Rightarrow (x^2yx^{-2})\gamma_1 = y\gamma_2.$$

$$\text{Let } y = e ; \quad (x^2yx^{-2})\gamma_1 = (x^2ex^{-2})\gamma_1 = (x^2x^{-2})\gamma_1 = e\gamma_1 = e\gamma_2 \Rightarrow e\gamma_1 = e\gamma_2.$$

$$\text{Let } y = x^m ; \quad (x^2yx^{-2})\gamma_1 = (x^2x^m x^{-2})\gamma_1 = x^{2+m-2}\gamma_1 = x^m\gamma_1 \Rightarrow x^m\gamma_2 \Rightarrow x^m\gamma_1 = x^m\gamma_2.$$

Corollary 2.1 *Let L be a C-loop. An autotopism of L can be constructed if there exists at least an $x \in L$ such that $x^2 \neq e$. The inverse can also be constructed.*

Proof

We need Lemma 2.1 and Theorem 2.3. If $x^2 = e$, then the autotopism is trivial. Since L is a C-loop, using Lemma 2.1 and Theorem 2.3, it will be noticed that $(\alpha_1 S_2, \beta_1 T_2, \gamma_1 \mathcal{R}_2) \in \text{Aut}(L)$ and $(\alpha_2 S_1, \beta_2 T_1, \gamma_2 \mathcal{R}_1) = (\alpha_1 S_2, \beta_1 T_2, \gamma_1 \mathcal{R}_2)^{-1}$. Hence the proof.

Remark 2.1 *If $x, y \in L, x \neq y$ such that $x^2 = y^2 \neq e$, then x and y will generate the same autotopism. If $x, y \in L$ such that $x^2 y^2 = e$, then the autotopism generated by x is the inverse of that generated by y .*

3 C-loops and Steiner loops

In [20], it was shown that every Steiner loop is a C-loop and Steiner loops are exactly inverse property loops of exponent two. Hence generally, C-loops are not Steiner loops. Recall that Steiner loops are totally symmetric loops, whence all parastrophes are equivalent to them. In this section, for a loop (L, \cdot) , if the triple $(U, V, W) \in \text{Aut}(L, \cdot)$ then, U , V and W will be referred to as the first, second and third components of the autotopism (U, V, W) . In the last section, the autotopisms $(S_1, T_1, \mathcal{R}_1)$ and $(S_2, T_2, \mathcal{R}_2)$ which shall be referred to as CS-autotopisms were used to construct the autotopisms $(\alpha_1 S_2, \beta_1 T_2, \gamma_1 \mathcal{R}_2)$ and $(\alpha_2 S_1, \beta_2 T_1, \gamma_2 \mathcal{R}_1)$. These four autotopisms are useful to us in this section. Particularly, the first component $\alpha_1 S_2$ and the second component $\beta_2 T_1$ are of paramount interest.

Theorem 3.1 *In every C-loop L , two of the parastrophes of L are equivalent(equal) to L , if and only if both the first and second components of the autotopisms*

$$(\alpha_1 S_2, \beta_1 T_2, \gamma_1 \mathcal{R}_2) \text{ and } (\alpha_2 S_1, \beta_2 T_1, \gamma_2 \mathcal{R}_1) \text{ respectively}$$

are equal to the identity map. Hence, the other three parastrophes are equivalent(equal) to L .

Proof

Using Theorem 2.3; $(x^2 y) \alpha_1 = y \alpha_2$ and $(y x^2) \beta_2 = y \beta_1 \Rightarrow (x x \cdot y) \alpha_1 = y \alpha_2$ and $(y \cdot x x) \beta_2 = y \beta_1 \Rightarrow (x \cdot x y) \alpha_1 = y \alpha_2$ and $(y x \cdot x) \beta_2 = y \beta_1 \Rightarrow (x \cdot x y) \alpha_1 \alpha_2^{-1} = y$ and $(y x \cdot x) \beta_2 \beta_1^{-1} = y \Rightarrow (x \cdot x y) \alpha_1 S_2 = y$ and $(y x \cdot x) \beta_2 T_1 = y$.

Let $x \cdot y = z$, then $(x \cdot z) \alpha_1 S_2 = y$. Let $x \theta y = z$ (θ replaces \cdot). Thus, $\alpha_1 S_2 : L \times L \rightarrow L$ is defined by $(x, z) \alpha_1 S_2 = (x \cdot z) \alpha_1 S_2 = y = x \theta^{-1} z \Leftrightarrow x \theta y = z$. (L, θ^{-1}) is a parastrophe of $(L, \theta) = (L, \cdot)$ by Definition 1.1. If $\alpha_1 S_2 = I$, then $(L, \theta) \equiv (L, \theta^{-1})$ i.e $(L, \theta) = (L, \theta^{-1})$.

Let $y \cdot x = t$, then $(t \cdot x) \beta_2 T_1 = y$. Thus, $\beta_2 T_1 : L \times L \rightarrow L$ is defined by $(t, x) \beta_2 T_1 = (t \cdot x) \beta_2 T_1 = y = t^{-1} \theta x \Leftrightarrow y \theta x = t$ (θ replaces \cdot). $(L, {}^{-1}\theta)$ is a parastrophe of $(L, \theta) = (L, \cdot)$ by Definition 1.1. If $\beta_2 T_1 = I$, then $(L, \theta) \equiv (L, {}^{-1}\theta)$ i.e $(L, \theta) = (L, {}^{-1}\theta)$.

Conversely, assume that $(L, \theta) \equiv (L, \theta^{-1})$ and $(L, \theta) \equiv (L, {}^{-1}\theta)$ i.e $(L, \theta) = (L, \theta^{-1})$ and $(L, \theta) = (L, {}^{-1}\theta)$ where (L, θ^{-1}) and $(L, {}^{-1}\theta)$ are as defined in Definition 1.1. Recall that ; $(x \cdot xy)\alpha_1 S_2 = y$ and $(yx \cdot x)\beta_2 T_1 = y$. Hence, if $z = x \cdot y$ and $t = y \cdot x$ then ; $(x \cdot xy)\alpha_1 S_2 = y$ and $(yx \cdot x)\beta_2 T_1 = y \Rightarrow (x \cdot z)\alpha_1 S_2 = y$ and $(t \cdot x)\beta_2 T_1 = y \Rightarrow (x\theta z)\alpha_1 S_2 = y$ and $(t\theta x)\beta_2 T_1 = y \Rightarrow (x\theta z)\alpha_1 S_2 = x\theta^{-1}z$ and $(t\theta x)\beta_2 T_1 = t^{-1}\theta x \Rightarrow \alpha_1 S_2 = I, \beta_2 T_1 = I$ because $(L, \theta) \equiv (L, \theta^{-1})$ and $(L, \theta) \equiv (L, {}^{-1}\theta)$.

The proof of the last part is as follows. Consider the definitions of the other three parastrophes in Definition 1.1.

$$\begin{aligned}
& \left(L, ({}^{-1}\theta)^* \right) = \left(L, {}^{-1}(\theta^{-1}) \right) \\
& = \left\{ x, y, z \in L : y {}^{-1}(\theta^{-1}) z = x \Leftrightarrow x\theta y = z \text{ where } \theta \text{ replaces } \cdot \right\}, \\
& \left(L, (\theta^{-1})^* \right) = \left(L, ({}^{-1}\theta)^{-1} \right) \\
& = \left\{ x, y, z \in L : z ({}^{-1}\theta)^{-1} x = y \Leftrightarrow x\theta y = z \text{ where } \theta \text{ replaces } \cdot \right\}, \\
& \left(L, \theta^* \right) = \left(L, ({}^{-1}(\theta^{-1}))^{-1} \right) \\
& = \left\{ x, y, z \in L : y {}^{-1}(\theta^{-1}) x = z \Leftrightarrow x\theta y = z \text{ where } \theta \text{ replaces } \cdot \right\}.
\end{aligned}$$

From the definitions above, it can be seen that the other three parastrophes can be derived from the first two. Hence, these three are equivalent to L since the first two are equivalent to L by the first part.

Corollary 3.1 *Both the first and second components of the autotopisms*

$$(\alpha_1 S_2, \beta_1 T_2, \gamma_1 \mathcal{R}_2) \text{ and } (\alpha_2 S_1, \beta_2 T_1, \gamma_2 \mathcal{R}_1) \text{ respectively}$$

of a C-loop (L, \cdot) are equal to the identity map if and only if L is a Steiner loop.

Proof

Let $\alpha_1 S_2 = I, \beta_2 T_1 = I$ then $S_1 = S_2, T_1 = T_2$. Whence, $x^2 = e$ and $x(xy) = y, (yx)x = y$ since a C-loop is alternative. This proves that L is a Steiner loop.

Conversely, if L is a Steiner loop, then L is of exponent 2. Recall that $\alpha_1 S_2 = L_x^{-2}$ and $\beta_2 T_1 = R_x^{-2}$. Hence, $\alpha_1 S_2 = L_e$ and $\beta_2 T_1 = R_e \Rightarrow \alpha_1 S_2 = I$ and $\beta_2 T_1 = I$.

Remark 3.1 *The result above generalizes the fact that Steiner loops are exactly C-loops that are of exponent 2.*

Theorem 3.2 *Let*

$$Q = \left\{ (S_i, T_i, \mathcal{R}_i), (S_{i+1}, T_{i+1}, \mathcal{R}_{i+1}), (S_i S_{i+1}, T_i T_{i+1}, \mathcal{R}_i \mathcal{R}_{i+1}) \in \text{Aut}(L) \mid \right.$$

$$(S_i, T_i, \mathcal{R}_i) = (A, B, C)(L_x^2, I, L_x^2), (S_{i+1}, T_{i+1}, \mathcal{R}_{i+1}) = (A, B, C)(I, R_x^2, R_x^2),$$

$$(A, B, C) \in \text{Aut}(L, \cdot) \Big\}_{i \in \mathbb{N}}$$

be a set of CS-autotopisms of a non-Steiner C-loop. Define

$$(Q, 3) = \left\{ \left\{ (S_i, T_i, \mathcal{R}_i), (S_{i+1}, T_{i+1}, \mathcal{R}_{i+1}), (S_i S_{i+1}, T_i T_{i+1}, \mathcal{R}_i \mathcal{R}_{i+1}) \right\} \mid i \in \mathbb{N} \right\}.$$

Then, $(Q, 3)$ is a Steiner triple system.

Proof

We shall show that Definition 1.2 is true for $(Q, 3)$.

1.

$$(S_i, T_i, \mathcal{R}_i), (S_{i+1}, T_{i+1}, \mathcal{R}_{i+1}), (S_i S_{i+1}, T_i T_{i+1}, \mathcal{R}_i \mathcal{R}_{i+1})$$

are distinct elements of $Q \forall i \in \mathbb{N}$.

2. For any $(S_i, T_i, \mathcal{R}_i), (S_{i+1}, T_{i+1}, \mathcal{R}_{i+1}) \in Q$, $(S_i, T_i, \mathcal{R}_i) \neq (S_{i+1}, T_{i+1}, \mathcal{R}_{i+1})$ or else L will become a Steiner loop. There exists a unique autotopism $(S_i S_{i+1}, T_i T_{i+1}, \mathcal{R}_i \mathcal{R}_{i+1}) \in \text{AUT}(L) \ni$

$$\{(S_i, T_i, \mathcal{R}_i), (S_{i+1}, T_{i+1}, \mathcal{R}_{i+1}), (S_i S_{i+1}, T_i T_{i+1}, \mathcal{R}_i \mathcal{R}_{i+1})\} \in (Q, 3)$$

is distinct.

Thus, by Definition 1.2, $(Q, 3)$ is a Steiner triple system.

3.1 Construction

Let us now consider the C-loop of order 12 whose multiplication table is shown in Table 1.

The construction of an autotopism of the finite C-loop whose bordered multiplication table is shown in Table 1 is now given below. Π_ρ denotes the right representation set of the loop.

By Theorem 2.3;

$$(x^2 y) \alpha_1 S_2 = y,$$

$$y x^2 = y \beta_1 T_2,$$

$$(x^2 y x^{-2}) \gamma_1 \mathcal{R}_2 = y.$$

Consider

$$(x^2 y) \alpha_1 S_2 = y$$

and fix $x = 4$.

$$\text{Let } y = 0, (4^2 \cdot 0) \alpha_1 S_2 = 0 \Rightarrow 2 \alpha_1 S_2 = 0.$$

$$\begin{aligned}
\text{Let } y = 1, & \left(4^2 \cdot 1\right) \alpha_1 S_2 = 1 \Rightarrow 0 \alpha_1 S_2 = 1. \\
\text{Let } y = 2, & \left(4^2 \cdot 2\right) \alpha_1 S_2 = 2 \Rightarrow 1 \alpha_1 S_2 = 2. \\
\text{Let } y = 3, & \left(4^2 \cdot 3\right) \alpha_1 S_2 = 3 \Rightarrow 5 \alpha_1 S_2 = 3. \\
\text{Let } y = 4, & \left(4^2 \cdot 4\right) \alpha_1 S_2 = 4 \Rightarrow 3 \alpha_1 S_2 = 4. \\
\text{Let } y = 5, & \left(4^2 \cdot 5\right) \alpha_1 S_2 = 5 \Rightarrow 4 \alpha_1 S_2 = 5. \\
\text{Let } y = 6, & \left(4^2 \cdot 6\right) \alpha_1 S_2 = 6 \Rightarrow 8 \alpha_1 S_2 = 6. \\
\text{Let } y = 7, & \left(4^2 \cdot 7\right) \alpha_1 S_2 = 7 \Rightarrow 6 \alpha_1 S_2 = 7. \\
\text{Let } y = 8, & \left(4^2 \cdot 8\right) \alpha_1 S_2 = 8 \Rightarrow 7 \alpha_1 S_2 = 8. \\
\text{Let } y = 9, & \left(4^2 \cdot 9\right) \alpha_1 S_2 = 9 \Rightarrow 11 \alpha_1 S_2 = 9. \\
\text{Let } y = 10, & \left(4^2 \cdot 10\right) \alpha_1 S_2 = 10 \Rightarrow 9 \alpha_1 S_2 = 10. \\
\text{Let } y = 11, & \left(4^2 \cdot 11\right) \alpha_1 S_2 = 11 \Rightarrow 10 \alpha_1 S_2 = 11.
\end{aligned}$$

Hence,

$$\alpha_1 S_2 = (0 \ 1 \ 2)(3 \ 4 \ 5)(6 \ 7 \ 8)(9 \ 10 \ 11) = \alpha^2 = R_1.$$

Consider

$$yx^2 = y\beta_1 T_2$$

·	0	1	2	3	4	5	6	7	8	9	10	11
0	0	1	2	3	4	5	6	7	8	9	10	11
1	1	2	0	4	5	3	7	8	6	10	11	9
2	2	0	1	5	3	4	8	6	7	11	9	10
3	3	4	5	0	1	2	9	10	11	6	7	8
4	4	5	3	1	2	0	10	11	9	7	8	6
5	5	3	4	2	0	1	11	9	10	8	6	7
6	6	7	8	10	11	9	0	1	2	5	3	4
7	7	8	6	11	9	10	1	2	0	3	4	5
8	8	6	7	9	10	11	2	0	1	4	5	3
9	9	10	11	8	6	7	3	4	5	2	0	1
10	10	11	9	6	7	8	4	5	3	0	1	2
11	11	9	10	7	8	6	5	3	4	1	2	0

Table 1: A non-associative C-loop of order 12

and fix $x = 4$.

$$\text{Let } y = 0, 0 \cdot 4^2 = 0\beta_1 T_2 \Rightarrow 2 = 0\beta_1 T_2.$$

$$\text{Let } y = 1, 1 \cdot 4^2 = 1\beta_1 T_2 \Rightarrow 0 = 1\beta_1 T_2.$$

$$\text{Let } y = 2, 2 \cdot 4^2 = 2\beta_1 T_2 \Rightarrow 1 = 2\beta_1 T_2.$$

$$\text{Let } y = 3, 3 \cdot 4^2 = 3\beta_1 T_2 \Rightarrow 5 = 3\beta_1 T_2.$$

$$\text{Let } y = 4, 4 \cdot 4^2 = 4\beta_1 T_2 \Rightarrow 3 = 4\beta_1 T_2.$$

$$\text{Let } y = 5, 5 \cdot 4^2 = 5\beta_1 T_2 \Rightarrow 4 = 5\beta_1 T_2.$$

$$\text{Let } y = 6, 6 \cdot 4^2 = 6\beta_1 T_2 \Rightarrow 8 = 6\beta_1 T_2.$$

$$\text{Let } y = 7, 7 \cdot 4^2 = 7\beta_1 T_2 \Rightarrow 6 = 7\beta_1 T_2.$$

$$\text{Let } y = 8, 8 \cdot 4^2 = 8\beta_1 T_2 \Rightarrow 7 = 8\beta_1 T_2.$$

$$\text{Let } y = 9, 9 \cdot 4^2 = 9\beta_1 T_2 \Rightarrow 11 = 9\beta_1 T_2.$$

$$\text{Let } y = 10, 10 \cdot 4^2 = 10\beta_1 T_2 \Rightarrow 9 = 10\beta_1 T_2.$$

$$\text{Let } y = 11, 11 \cdot 4^2 = 11\beta_1 T_2 \Rightarrow 10 = 11\beta_1 T_2.$$

Hence,

$$\beta_1 T_2 = (0 \ 2 \ 1)(3 \ 5 \ 4)(6 \ 8 \ 7)(9 \ 11 \ 10) = \alpha^{-2} = R_2.$$

Consider

$$(x^2 y x^{-2}) \gamma_1 \mathcal{R}_2 = y$$

and fix $x = 4$.

$$\text{Let } y = 0, \left(4^2 \cdot 0 \cdot 4^2\right) \gamma_1 \mathcal{R}_2 = 0 \Rightarrow 0 \gamma_1 \mathcal{R}_2 = 0.$$

$$\text{Let } y = 1, \left(4^2 \cdot 1 \cdot 4^2\right) \gamma_1 \mathcal{R}_2 = 1 \Rightarrow 1 \gamma_1 \mathcal{R}_2 = 1.$$

$$\text{Let } y = 2, \left(4^2 \cdot 2 \cdot 4^2\right) \gamma_1 \mathcal{R}_2 = 2 \Rightarrow 2 \gamma_1 \mathcal{R}_2 = 2.$$

$$\text{Let } y = 3, \left(4^2 \cdot 3 \cdot 4^2\right) \gamma_1 \mathcal{R}_2 = 3 \Rightarrow 3 \gamma_1 \mathcal{R}_2 = 3.$$

$$\text{Let } y = 4, \left(4^2 \cdot 4 \cdot 4^2\right) \gamma_1 \mathcal{R}_2 = 4 \Rightarrow 4 \gamma_1 \mathcal{R}_2 = 4.$$

$$\text{Let } y = 5, \left(4^2 \cdot 5 \cdot 4^2\right) \gamma_1 \mathcal{R}_2 = 5 \Rightarrow 5 \gamma_1 \mathcal{R}_2 = 5.$$

$$\text{Let } y = 6, \left(4^2 \cdot 6 \cdot 4^2\right) \gamma_1 \mathcal{R}_2 = 6 \Rightarrow 6 \gamma_1 \mathcal{R}_2 = 6.$$

$$\text{Let } y = 7, \left(4^2 \cdot 7 \cdot 4^2\right) \gamma_1 \mathcal{R}_2 = 7 \Rightarrow 7 \gamma_1 \mathcal{R}_2 = 7.$$

$$\text{Let } y = 8, \left(4^2 \cdot 8 \cdot 4^2\right) \gamma_1 \mathcal{R}_2 = 8 \Rightarrow 8 \gamma_1 \mathcal{R}_2 = 8.$$

$$\text{Let } y = 9, \left(4^2 \cdot 9 \cdot 4^2\right) \gamma_1 \mathcal{R}_2 = 9 \Rightarrow 9 \gamma_1 \mathcal{R}_2 = 9.$$

$$\text{Let } y = 10, \left(4^2 \cdot 10 \cdot 4^2\right) \gamma_1 \mathcal{R}_2 = 10 \Rightarrow 10 \gamma_1 \mathcal{R}_2 = 10.$$

$$\text{Let } y = 11, \left(4^2 \cdot 11 \cdot 4^2\right) \gamma_1 \mathcal{R}_2 = 11 \Rightarrow 11 \gamma_1 \mathcal{R}_2 = 11.$$

Hence,

$$\gamma_1 \mathcal{R}_2 = (0)(1)(2)(3)(4)(5)(6)(7)(8)(9)(10)(11) = I = R_0.$$

So,

$$\alpha_1 S_2 = (0 \ 1 \ 2)(3 \ 4 \ 5)(6 \ 7 \ 8)(9 \ 10 \ 11) = \alpha^2 = R_1,$$

$$\beta_1 T_2 = (0 \ 2 \ 1)(3 \ 5 \ 4)(6 \ 8 \ 7)(9 \ 11 \ 10) = \alpha^{-2} = R_2,$$

$$\gamma_1 \mathcal{R}_2 = (0)(1)(2)(3)(4)(5)(6)(7)(8)(9)(10)(11) = I = R_0.$$

Therefore,

$$(\alpha_1 S_2, \beta_1 T_2, \gamma_1 \mathcal{R}_2) = (\alpha^2, \alpha^{-2}, I) = (R_1, R_2, R_0) = (R_1, R_2, I) \in \text{Aut}(L), \alpha = R_{10} \in \Pi_\rho.$$

This is a principal autotopism.

Also, we can construct an autotopism for the C-loop whose unbordered multiplication table is in [20] by taking the steps of the construction above. Fix $x = 9$ and let $\mu = (0 \ 13 \ 5 \ 14)(1 \ 15 \ 4 \ 12)(2 \ 9 \ 10 \ 8)(3 \ 7 \ 11 \ 6) = R_{13}$. Then,

$$\alpha_1 S_2 = (0 \ 5)(1 \ 4)(2 \ 10)(3 \ 11)(6 \ 7)(8 \ 9)(12 \ 15)(13 \ 14) = \mu^2 = R_5,$$

$$\beta_1 T_2 = \alpha_1 S_2 = R_5,$$

$$\gamma_1 \mathcal{R}_2 = I = R_0.$$

Thus, $(\mu^2, \mu^2, I) = (R_5, R_5, R_0) \in \text{Aut}(L), \mu = R_{13} \in \Pi_\rho$.

References

- [1] R. Artzy (1963), *Isotopy and Parastrophy of Quasigroups*, Proc. Amer. Math. Soc. 14, 3, 429–431.
- [2] A. Beg (1977), *A theorem on C-loops*, Kyungpook Math. J. 17(1), 91–94.
- [3] A. Beg (1980), *On LC-, RC-, and C-loops*, Kyungpook Math. J. 20(2), 211–215.
- [4] V. D. Belousov (1965), *Systems of quasigroups with generalised identities*, Usp. Mat. Nauk. 20, 1(121), 75–146.
- [5] R. Capodaglio Di Cocco (1993), *On Isotopism and Pseudo-Automorphism of the loops*, Est.da : Boll. Uni. Mat. Italiana[7-A], 199–205.
- [6] O. Chein, H. O. Pflugfelder and J. D. H. Smith (1990), *Quasigroups and Loops : Theory and Applications*, Heldermann Verlag, 568pp.

- [7] J. Dénes and A. D. Keedwell (1974), *Latin square and their applications*, Academic Press, New York, London.
- [8] J. Duplak (2000), *A parastrophic equivalence in quasigroups*, Quasigroups and Related Systems 7, 7–14.
- [9] F. Fenyves (1969), *Extra Loops II*, Publ. Math. Debrecen 16, 187–192.
- [10] R. A. Fisher and F. Yates (1934), *The 6×6 Latin squares*, Proc. Camb. Philos. Soc. 30, 429–507.
- [11] F. E. Frank, E. Bennett and H. Zhang (2004), *Latin Squares with Self-Orthogonal Conjugates*, Discrete Mathematics, 284(1-3), 45–55.
- [12] E. G. Goodaire, E. Jespers and C. P. Milies (1996), *Alternative Loop Rings*, NHMS(184) Elsevier.
- [13] M. K. Kinyon, K. Kunen, J. D. Phillips (2002), *A generalization of Moufang and Steiner loops*, Alg. Univer. 48,1, 81–101.
- [14] M. K. Kinyon, J. D. Phillips and P. Vojtěchovský (2005), *Loops of Bol-Moufang type with a subgroup of index two*, Bul. Acad. Stiinte Repub. Mold. Mat. 3(49), 71–87.
- [15] M. K. Kinyon, J. D. Phillips and P. Vojtěchovský, *C-loops : Extensions and construction*, J. Alg. and Applica. (to appear).
- [16] C. C. Lindner and D. Steedley (1975), *On the number of conjugates of a quasigroup*, Journal Algebra Universalis, 5(1), 191–196.
- [17] H. O. Pflugfelder (1990), *Quasigroups and Loops : Introduction*, Heldermann Verlag, Sigma series in Pure Mathematics : 7.
- [18] J. D. Phillips and P. Vojtěchovský (2005), *The varieties of loops of Bol-Moufang type*, Alg. Univer. 3(54), 259–383.
- [19] J. D. Phillips and P. Vojtěchovský (2005), *The varieties of quasigroups of Bol-Moufang type : An equational approach*, J. Alg. 293, 17–33.
- [20] J. D. Phillips and P. Vojtěchovský (2006), *On C-loops*, Publ. Math. Debrecen. 68, 1-2, 115–137.
- [21] V. S. Ramamurthi and A. R. T. Solarin (1988), *On finite right central loops*, Publ. Math. Debrecen, 35, 260–264.
- [22] A. Sade (1959), *Quasigroupes parastrophiques*, Math. Nachr. 20, 73–106.
- [23] K. K. Shchukin and V. V. Gushan (2004), *A representation of parastrophs of loops and quasigroups*, Journal Discrete Mathematics and Applications, 14(5), 535–542.

- [24] S. K. Stein (1956), *Foundation of quasigroups*, Proc. Nat. Acad. Sci. 42, 545–545.
- [25] S. K. Stein (1957), *On the foundation of quasigroups*, Trans. Amer. Math. Soc. 85, 228–256.